

The Bifurcation of the Topological Structure in the Sunspot's Electric Topological Current with Locally Gauge-invariant Maxwell-Chern-Simons Term

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(August 27, 1998)

Abstract

The topological structure of the electric topological current of the locally gauge invariant Maxwell-Chern-Simons Model and its bifurcation is studied. The electric topological charge is quantized in term of winding number. The Hopf indices and Brouwer degree labeled the local topological structure of the electric topological current. Using Φ -mapping method and implicit theory, the electric topological current is found generating or annihilating at the limit points and splitting or merging at the bifurcate points. The total electric charge holds invariant during the evolution.

I. INTRODUCTION

In order to search for the model of sunspots to generate mass to the gauge field, Saniga, and Klacka (1992a-d etc.) had given the so-called Abelian Higgs (AH) model of sunspots, which are regarded as topological AH magnetic vortices. The central idea of the model is that the confinement of sunspot's magnetic field into a finite-dimensional, sharply-bounded flux tube is due to the so-called Higgs mechanism; a Higgs field spontaneously breaks an original U(1)-symmetry of a corresponding field configuration giving thus rise to a non-zero mass of photon. As another crucial point in such approach it turned out to be the assumption of cylindrical symmetry of sunspots. However there exist another interesting possibility of generating mass to the gauge field that might be seen as alternative to the standard Higgs mechanism. Namely, a photon can acquire a mass if the Lagrangian density contains, in addition to the classical Maxwellian term, also the so-called Chern-Simons (CS) one (see,

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e.g. Moradi, 1992). Although the resulting Lagrangian density leads to gauge-invariant equations of motion it is not itself a gauge-invariant quantity. Later Saniga (1993) brought the Lagrangian density to a gauge-invariant form and showed that it admits topological vortices. These vortices were then examined in some detail and they are shown to carry both the quantized magnetic flux and electric charge; but also endowed with a non-zero angular momentum they might thus serve as a starting point for the description of rotating sunspots.

The aim of the present paper is to reveal the topological properties of the electric topological current which is given in the locally gauge-invariant Chern-Simons model of the sunspot. By regarding the complex scalar field ζ , induced in the locally gauge-invariant MCS theory, as the complex representation of a two-dimensional vector field $\vec{\zeta} = (\zeta^1, \zeta^2)$ with ζ^1 and ζ^2 being the real and imaginary part of ζ , the electric current is expressed as a topological current. Using the ϕ -mapping method and generalized function theory, the structure of the electric topological current is proved taking the same form as the classical current density in usual hydrodynamics, in which the point-like particle with topological charge $\frac{\Theta\pi}{e}\beta_l\eta_l$, which is just the winding number of ζ at its zeroes, are called topological electric charge. The locally topological structure of the electric topological current is detailed, which is quantized by the winding number of the integral surface and the field function ζ and is labeled by the Hopf indices and Brouwer degrees of ζ . Further, by imposing the implicit theory, we show that there exist the crucial case of branch process in the electric topological current. It will be seen that the electric topological current is unstable and splitting (or merging) at the points where the corresponding Jacobian determinant $D(\frac{\zeta}{u})$ of field function ζ vanishes. The branch processes at the limit points as well as at the bifurcation point in the electric topological current are studied systemically. The electric topological current is found generating or annihilating at the limit points and splitting or merging at the bifurcation points. The former relates to the origin of the electric topological current. For the electric topological current is identically conserved, the total electric charge holds invariant during the process.

II. MAXWELL-CHERN-SIMONS ELECTROMAGNETISM AND ELECTRIC TOPOLOGICAL CURRENT DENSITY

Under the Maxwell-Chern-Simons configuration of field in the (2+1)-dimensional Minkowski space-time, the field configuration represented by the following Lagrangian density

$$\mathcal{L}^\odot = -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} + \frac{\Theta}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho \quad (1)$$

where $F_{\rho\sigma} \equiv \partial_\rho A_\sigma - \partial_\sigma A_\rho$, Θ is a constant, and the space-time is endowed with the metric tensor of the signature $(+ - -)$. In order to generalize (1) in a way to exhibit a gauge-invariant behavior, the Lagrangian is modified in the way

$$\mathcal{L}_G^\odot = -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} + \frac{\Theta}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}[A_\rho - \frac{i}{2e\zeta\zeta^*}\frac{\partial}{\partial x^\mu}(\zeta^*\frac{\partial\zeta}{\partial x^\nu} - \zeta\frac{\partial\zeta^*}{\partial x^\nu})]$$

with ζ having the following gauge transformation rule

$$\zeta \rightarrow \tilde{\zeta} = e^{i\alpha}\zeta, \quad \zeta^* \rightarrow \tilde{\zeta}^* = e^{-i\alpha}\zeta^*;$$

no other constraints are imposed on the scalar, complex-valued field ζ except the requirement of its single-valuedness. Using the modified Lagrangian, one can get the electric topological current density j^ρ of matter field, i.e.

$$j^\rho = -\frac{i\Theta}{4e}\epsilon^{\mu\nu\rho}\frac{\partial}{\partial x^\mu}\left(\frac{\zeta^*}{\zeta\zeta^*}\frac{\partial\zeta}{\partial x^\nu} - \frac{\zeta}{\zeta\zeta^*}\frac{\partial\zeta^*}{\partial x^\nu}\right) \quad (2)$$

This expression is of a topological nature since it is conserved automatically

$$\frac{\partial j^\rho}{\partial x^\rho} = -\frac{i\Theta}{4e}\epsilon^{\mu\nu\rho}\frac{\partial^2}{\partial x^\rho\partial x^\mu}\left(\frac{\zeta^*}{\zeta\zeta^*}\frac{\partial\zeta}{\partial x^\nu} - \frac{\zeta}{\zeta\zeta^*}\frac{\partial\zeta^*}{\partial x^\nu}\right) \equiv 0,$$

due to a complete antisymmetry of the Levi-Civita symbol. Hence the spot's total electric charge Q^{el}

$$Q^{el} \equiv \int j^\rho dS_\rho = -\int \frac{i\Theta}{4e}\epsilon^{\mu\nu\rho}\frac{\partial}{\partial x^\mu}\left(\frac{\zeta^*}{\zeta\zeta^*}\frac{\partial\zeta}{\partial x^\nu} - \frac{\zeta}{\zeta\zeta^*}\frac{\partial\zeta^*}{\partial x^\nu}\right)dS_\rho \quad (3)$$

is a purely topological quantity in exactly the same way as postulated by Kovner and Rosenstein (1992) for electric charge in quantum electrodynamics.

Because the complex scalar field ζ can be denoted by

$$\zeta = \zeta^1 + i\zeta^2, \quad (4)$$

we regard ζ as the complex representation of the vector field

$$\vec{\zeta} = (\zeta^1, \zeta^2). \quad (5)$$

Then we can define an unit vector

$$n^a = \zeta^a / \|\zeta\|, \quad \|\zeta\| = \sqrt{\zeta^a \zeta^a}, \quad (6)$$

satisfying

$$n^a n^a = 1, \quad a = 1, 2.$$

In fact \vec{n} is identified as a section of the sphere bundle over the space-time, i.e. \vec{n} is a section of the $U(1)$ line bundle. It is obvious that the zeroes of $\vec{\zeta}$ are just the singular points of \vec{n} . Now j^ρ can be written as

$$j^\rho = \frac{\Theta}{2e}\epsilon^{\mu\nu\rho}\epsilon^{ab}\partial_\mu n^a \partial_\nu n^b \quad (7)$$

This formula of j^ρ takes the same form as the topological current of the torsion in Duan, Zhang and Feng's paper (1994). Hence we call j^ρ electric topological current.

Substituting (7) into (3), the spot's total electric charge is

$$Q^{el} = \int_S \frac{\Theta}{2e}\epsilon^{\mu\nu\rho}\epsilon^{ab}\partial_\mu n^a \partial_\nu n^b dS_\rho \quad (8)$$

III. QUANTIZATION AND LOCAL TOPOLOGICAL PROPERTIES OF THE SUNSPOT'S ELECTRIC TOPOLOGICAL CURRENT

To study the quantization and local topological properties of the electric topological current, let us choose coordinates $y = (u^1, u^2, \tau)$ of the space-time such that $u = (u^1, u^2)$ be the intrinsic coordinate on S . For the coordinate component τ does not belong to S . Then

$$Q^{el} = \int_S \frac{\Theta}{2e} \epsilon^{ij} \epsilon^{ab} \frac{\partial n^a}{\partial u^i} \frac{\partial n^b}{\partial u^j} dS \quad (9)$$

where $i, j = 1, 2$ and

$$dS = \frac{1}{2} \epsilon^{\nu\rho} \epsilon_{ij} \frac{\partial u^i}{\partial x^\mu} \frac{\partial u^j}{\partial x^\nu} dS_\rho = du^1 du^2 \quad (10)$$

is the element of the surface S . Notice

$$\epsilon^{ij} \epsilon^{ab} \frac{\partial n^a}{\partial u^i} \frac{\partial n^b}{\partial u^j} = \epsilon^{ab} \epsilon^{ij} \frac{\partial}{\partial \zeta^c} \frac{\partial}{\partial \zeta^a} \ln \|\zeta\| \cdot \frac{\partial \zeta^c}{\partial u^i} \frac{\partial \zeta^b}{\partial u^j} dS_\rho. \quad (11)$$

Define the Jacobian determinant $D(\frac{\zeta}{u})$ as

$$\epsilon_{ij} D(\frac{\zeta}{u}) = \epsilon^{ab} \frac{\partial \zeta^a}{\partial u^i} \frac{\partial \zeta^b}{\partial u^j}. \quad (12)$$

Using of Laplacian relation in ζ -space

$$\frac{\partial^2}{\partial \zeta^a \partial \zeta^a} \ln \|\zeta\| = 2\pi \delta^2(\zeta), \quad (13)$$

we rewrite the equation (8) in a compact form

$$Q^{el} = \frac{\Theta\pi}{e} \int D(\frac{\zeta}{k}) \delta^2(\zeta) dS \quad (14)$$

The equation (14) shows that only those points, on which $\zeta = 0$, contribute to Q^{el} .

Suppose that the vector field $\zeta^a(x)$ possesses N zeroes on S and let the l th zero be $x = z_l$

$$\zeta^a(z_l) = 0 \quad (15)$$

According to the deduction of Duan and Liu (1988) and the implicit function theorem (see e.g. Edourd, 1904), the solutions of $\zeta(u^1, u^2, \tau) = 0$ can be expressed in terms of $u = (u^1, u^2)$ as

$$u^i = z^i(\tau), \quad i = 1, 2 \quad (16)$$

and

$$\zeta^a(z_l^1(\tau), z_l^2(\tau), \tau) \equiv 0, \quad (17)$$

where the subscript $l = 1, 2, \dots, N$ represents the l th zero of ζ , i.e.

$$\zeta^a(z_l^i) = 0, \quad (18)$$

It is easy to get the following formula from the ordinary theory of δ -function that

$$\delta^2(\zeta) D\left(\frac{\zeta}{u}\right) = \sum_{l=1}^N \beta_l \eta_l \delta(u - z_l). \quad (19)$$

The positive integer β_l is called the Hopf index of map $x \rightarrow \zeta$ (see, e.g. Milnor, 1965; Dubrovin, 1985; Duan 1979), which means that when the point x covers the neighborhood of the zero $x = z_l$ once, the function ζ^i covers the corresponding region in ζ -space β_l times. And

$$\eta_l = \frac{D(\frac{\zeta}{u})}{|D(\frac{\zeta}{u})|} \Big|_{x=z_l} = \pm 1, \quad (20)$$

is called the Brouwer degree of map $x \rightarrow \zeta$ (see Duan 1979, 1990). That the Hopf indices be integers is due to the single-valueness of ζ . Substituting this expansion of $\delta^2(\zeta)$ into (14), Q^{el} is quantized in the topological level as

$$Q^{el} = \frac{\Theta\pi}{e} \int \sum_{l=1}^N \beta_l \eta_l \delta^2(u - z_l) dS = \frac{\Theta\pi}{e} \sum_{l=1}^N \beta_l \eta_l. \quad (21)$$

Hence, the total electric charge of the MCS sunspot is quantized, i.e. it is composed of an integer number of the ‘universal’ unit Q_0^{el} . Furthermore, from (21) we see that this total electric charge is composed by many independent quantized charges which locate at the zeroes of the complex scalar field ζ and the topological structure of the electric density is labeled by Hopf index β_i and Brouwer degree η_i .

On another hand, the winding number of the surface S and the mapping ζ is defined as (Victor and Alan, 1974)

$$W = \frac{1}{2\pi} \int_S \epsilon^{ij} \epsilon^{ab} \frac{\partial n^a}{\partial u^i} \frac{\partial n^b}{\partial u^j} dS. \quad (22)$$

which is equal to the number of times S encloses (or, wraps around) the point $\zeta = 0$. Hence, the total electric charge is quantized by the winding number

$$Q^{el} = W \frac{\Theta\pi}{e}. \quad (23)$$

The winding number W of the surface S can be interpreted or, indeed, defined as the degree of the mapping ζ onto S . By (14) we have

$$\begin{aligned} Q^{el} &= \frac{\Theta\pi}{e} \int_S \delta(\zeta) D\left(\frac{\zeta}{u}\right) du^1 du^2 \\ &= \frac{\Theta\pi}{e} \deg \zeta \int_{\zeta(S)} \delta(\zeta) d\zeta^1 d\zeta^2 \\ &= \frac{\Theta\pi}{e} \deg \zeta \end{aligned}$$

where $\deg \zeta$ is the degree of map $\zeta : S \rightarrow \zeta(S)$. Compared above equation with (23), it shows the degree of map $\zeta : S \rightarrow \zeta(S)$ is just the winding number W of the surface S and the mapping ζ , i.e.

$$W = \deg \zeta$$

Then the total electric topological charge is

$$Q^{el} = W Q_0^{el} = \deg \zeta Q_0^{el}. \quad (24)$$

Divide S by

$$S = \sum_{l=1}^N S_l$$

and S_l includes only one zero z_l of ζ , i.e. $z_l \in S_l$. Then The winding number of the surface S_l and the mapping ζ is

$$W_l = \frac{1}{2\pi} \int_{S_l} \epsilon^{ij} \epsilon^{ab} \frac{\partial n^a}{\partial u^i} \frac{\partial n^b}{\partial u^j} dS$$

which is equal to the number of times S_l encloses (or, wraps around) the point $u = z_l$. It is easy to see that

$$W = \sum_{l=1}^N W_l$$

and

$$\beta_l = |W_l| \quad \eta_l = \text{sign} W_l.$$

Then

$$Q^{el} = Q_0^{el} \sum_{l=1}^N W_l = Q_0^{el} \sum_{l=1}^N \beta_l \eta_l.$$

Denote the sum of β_l with $\eta_l = 1$ and $\eta_l = -1$ as W^+ and W^- respectively, the total electric charge can be rewritten as

$$Q^{el} = \frac{\Theta\pi}{e} (W^+ - W^-), \quad (25)$$

which reveal the contributions of electric topological current with positive or negative charges. Since the zero of ζ at $u = z_l$ gives a vortexlike structure, the expression (3) gives the vorticity in space-time. So W^+ and W^- denote the vorticity of vortex and antivortex. Therefore the vortex and antivortex classify positive or negative charges.

IV. THE BIFURCATION OF THE TOPOLOGICAL STRUCTURE OF ELECTRIC TOPOLOGICAL CURRENT

Using the intrinsic coordinates, redefine the topological electronic current as

$$j^\rho = \frac{\Theta}{2e} \epsilon^{\mu\nu\rho} \epsilon^{ab} \frac{\partial n^a}{\partial y^\mu} \frac{\partial n^b}{\partial y^\nu} \quad (26)$$

From (17) we can prove that the general velocity of the l th zero

$$V^j := \frac{dz_l^j}{d\tau} = \frac{D^j(\zeta/u)}{D(\zeta/u)} \Big|_{u=z_l} \quad V^0 = 1. \quad (27)$$

Then the electric topological current j^ρ can be written as the form of the current density of the system of l classical point particles with topological charge $Q_l^{el} = \eta_l \beta_l Q_0^{el}$ moving in the $(2+1)$ -dimensional space-time

$$j^i = \sum_{l=1}^N Q_l^{el} \delta(u - z_l(\tau)) \frac{du_l^i}{d\tau}, \quad j^0 = \sum_{l=1}^N Q_l^{el} \delta(k - z_l(\tau)) \quad (28)$$

The total charge of the system is

$$Q^{el} = \int_S j^\rho dS_\rho = \int_S j^0 dS = Q_0^{el} \sum_{l=1}^N \beta_l \eta_l, \quad (29)$$

Therefore we get a concise expression for the topological current,

$$j^i = j^0 \frac{D^i(\zeta/u)}{D(\zeta/u)} = j^0 V^i = Q_0^{el} \sum_{l=1}^N \beta_l \eta_l V^i, \quad (30)$$

which takes the same form as the current density in hydrodynamics. Expressions (28) and (30) give the topological structure of the electric topological current, which is characterized by the Brouwer degrees and Hopf indices. In our theory the point-like particles with topological charges $Q_l^{el} = Q_0^{el} \beta_l \eta_l$ ($l = 1, 2, \dots, N$) are called topological particles, the charges of which are topologically quantized and these particles are just located at the zeros of $\zeta(u)$, i.e. the singularities of the unit vector $n(u)$.

The above discussion is based on the condition that the Jacobian

$$D\left(\frac{\zeta}{u}\right) \Big|_{z_l} \neq 0. \quad (31)$$

When $D\left(\frac{\zeta}{u}\right) \Big|_{z_l} = 0$, it is shown that there exist the crucial case of branch process. There are two kinds of branch points namely limit points and bifurcation points, which will be discussed in detail in the follows.

Firstly we study the case when the zeros of $\vec{\zeta}$ include some limit points. The limit points are determined by

$$\begin{cases} \zeta^1(u^1, u^2, \tau) = 0 \\ \zeta^2(u^1, u^2, \tau) = 0 \\ \zeta^3(u^1, u^2, \tau) = D(\frac{\zeta}{u}) = 0 \end{cases} \quad (32)$$

and

$$D^A(\frac{\zeta}{u})|_{(z_l, \tau^*)} \neq 0, \quad A = 1, 2 \quad (33)$$

where we denote the limit points as (z_l, τ^*) . Since the usual implicit function theorem is of no use when the Jacobian determinant $D(\frac{\zeta}{u})|_{z_l} = 0$. For the purpose of using the implicit function theorem to study the branch properties of electric topological current at the limit points, we use the Jacobian $D^1(\frac{\zeta}{u})$ instead of $D(\frac{\zeta}{u})$ to search for the solutions of $\zeta = 0$. This means we have replaced u^1 by τ . For clarity we rewrite the first two equations of (32) as

$$\zeta^a(\tau, u^2, u^1) = 0, \quad a = 1, 2. \quad (34)$$

Taking account of (33) and using the implicit function theorem, we have a unique solution of the equations (34) in the neighborhood of the limit point (z_l, τ^*)

$$\tau = \tau(u^1), \quad u^2 = u^2(u^1) \quad (35)$$

with $\tau^* = \tau(z_l^1)$. In order to show the behavior of the electric topological current at the limit points, we will investigate the Taylor expansion of (35) in the neighborhood of (z_l, τ^*) . In the present case, from (33) and the last equation of (32), we get

$$\frac{du^1}{d\tau}|_{(z_l, \tau^*)} = \frac{D^1(\frac{\zeta}{u})}{D(\frac{\zeta}{u})}|_{(z_l, \tau^*)} = \infty$$

i.e.

$$\frac{d\tau}{du^1}|_{(z_l, \tau^*)} = 0.$$

Then, the Taylor expansion of $\tau = \tau(u^1)$ at the limit point (z_l, τ^*) is

$$\begin{aligned} \tau &= \tau(u_l^1) + \frac{d\tau}{du^1}|_{(z_l, \tau^*)}(u^1 - z_l^1) + \frac{1}{2} \frac{d^2\tau}{(du^1)^2}|_{(z_l, \tau^*)}(u^1 - z_l^1)^2 \\ &= \tau^* + \frac{1}{2} \frac{d^2\tau}{(du^1)^2}|_{(z_l, \tau^*)}(u^1 - z_l^1)^2. \end{aligned} \quad (36)$$

Therefore

$$\tau - \tau^* = \frac{1}{2} \frac{d^2\tau}{(du^1)^2}|_{(z_l, \tau^*)}(u^1 - z_l^1)^2 \quad (37)$$

which is a parabola in $u^1 - \tau$ plane. From (37) we can obtain two solutions $u_1^1(\tau)$ and $u_2^1(\tau)$, which give the branch solutions of electric topological current at the limit points. If

$\frac{d^2\tau}{(du^1)^2}|_{(z_l, \tau^*)} > 0$, we have the branch solutions for $\tau > \tau^*$, otherwise, we have the branch solutions for $\tau < \tau^*$. The former related to the origin of the topological electric charges and the later is related the annihilate of the topological electric charges.

Since the electric topological current is identically conserved, the topological quantum numbers of these two generated or annihilated topological electric charges must be opposite at the limit point, i.e.

$$\beta_1 \eta_1 Q_0^{el} + \beta_2 \eta_2 Q_0^{el} = 0,$$

which shows that the limit points do not contribute to the total electric charge.

Now, let us turn to consider the other case, in which the restrictions are

$$D\left(\frac{\zeta}{u}\right)|_{(z_l, \tau^*)} = 0, \quad D^1\left(\frac{\zeta}{u}\right)|_{(z_l, \tau^*)} = 0. \quad (38)$$

These two restrictive conditions will lead to an important fact that the function relationship between τ and u^1 is not unique in the neighborhood of (τ^*, z_l) . In our electric topological current theory this fact is easily seen from one of the Eqs.(27)

$$V^1 = \frac{du^1}{d\tau} = \frac{D^1\left(\frac{\zeta}{u}\right)}{D\left(\frac{\zeta}{u}\right)}|_{(z_l, \tau^*)} \quad (39)$$

which under (38) directly shows that the direction of the integral curve of (39) is indefinite at (z_l, τ^*) . Therefore the very point (z_l, τ^*) is called a bifurcation point of the electric topological current. With the aim of finding the different directions of all branch curves at the bifurcation point, we suppose that

$$\frac{\partial \zeta^1}{\partial u^2}|_{(z_l, \tau^*)} \neq 0. \quad (40)$$

From $\zeta^1(u^1, u^2, \tau) = 0$, the implicit function theorem says that there exists one and only one function relationship

$$u^2 = u^2(u^1, \tau). \quad (41)$$

Substituting (41) into ζ^1 , we have

$$\zeta^1(u^1, u^2(u^1, \tau), \tau) \equiv 0$$

which gives

$$\begin{aligned} \frac{\partial \zeta^1}{\partial u^2} f_1^2 &= -\frac{\partial \zeta^1}{\partial u^1}, & \frac{\partial \zeta^1}{\partial u^2} f_\tau^2 &= -\frac{\partial \zeta^1}{\partial \tau}, \\ \frac{\partial \zeta^1}{\partial k^2} f_{11}^2 &= -2 \frac{\partial^2 \zeta^1}{\partial k^2 \partial k^1} f_1^2 - \frac{\partial^2 \zeta^1}{(\partial u^2)^2} (f_1^2)^2 - \frac{\partial^2 \zeta^1}{(\partial u^1)^2}, \\ \frac{\partial \zeta^1}{\partial u^2} f_{1\tau}^2 &= -\frac{\partial^2 \zeta^1}{\partial u^2 \partial \tau} f_1^2 - \frac{\partial^2 \zeta^1}{\partial u^2 \partial u^1} f_\tau^2 - \frac{\partial^2 \zeta^1}{(\partial k^2)^2} f_\tau^2 f_1^2 - \frac{\partial^2 \zeta^1}{\partial k^1 \partial \tau}, \end{aligned} \quad (42)$$

$$\frac{\partial \zeta^1}{\partial u^2} f_{\tau\tau}^2 = -2 \frac{\partial^2 \zeta^1}{\partial u^2 \partial \tau} f_\tau^2 - \frac{\partial^2 \zeta^1}{(\partial u^2)^2} (f_\tau^2)^2 - \frac{\partial^2 \zeta^1}{\partial \tau^2},$$

where the partial derivatives is

$$f_1^2 = \frac{\partial u^2}{\partial u^1}, \quad f_\tau^2 = \frac{\partial u^2}{\partial \tau}, \quad f_{11}^2 = \frac{\partial^2 u^2}{(\partial u^1)^2},$$

$$f_{1\tau}^2 = \frac{\partial^2 u^2}{\partial u^1 \partial \tau}, \quad f_{\tau\tau}^2 = \frac{\partial^2 u^2}{\partial \tau^2}.$$

From these expressions it is easy to calculate the values of $f_1^2, f_\tau^2, f_{11}^2, f_{1\tau}^2$ and $f_{\tau\tau}^2$ at (τ^*, z_l) .

In order to explore the behavior of the electric topological current at the bifurcation points, let us investigate the Taylor expansion of

$$f(u^1, \tau) = \zeta^2(u^1, u^2(u^1, \tau), \tau) \quad (43)$$

in the neighborhood of (z_l, τ^*) , which according to the Eqs.(32) must vanish at the bifurcation point, i.e.

$$f(z_l, \tau^*) = 0. \quad (44)$$

From (43), the first order partial derivatives of $f(u^1, \tau)$ with respect to u^1 and τ can be expressed by

$$\frac{\partial F}{\partial u^1} = \frac{\partial \zeta^2}{\partial u^1} + \frac{\partial \zeta^2}{\partial u^2} f_1^2, \quad \frac{\partial F}{\partial \tau} = \frac{\partial \zeta^2}{\partial \tau} + \frac{\partial \zeta^2}{\partial u^2} f_\tau^2. \quad (45)$$

By making use of (42), (45) and Cramer's rule, it is easy to prove that the two restrictive conditions (38) can be rewritten as

$$D\left(\frac{\zeta}{u}\right)|_{(z_l, \tau^*)} = \left(\frac{\partial f}{\partial u^1} \frac{\partial \zeta^1}{\partial u^2}\right)|_{(z_l, \tau^*)} = 0,$$

$$D^1\left(\frac{\zeta}{u}\right)|_{(z_l, \tau^*)} = \left(\frac{\partial f}{\partial \tau} \frac{\partial \zeta^1}{\partial u^2}\right)|_{(z_l, \tau^*)} = 0,$$

which give

$$\frac{\partial f}{\partial u^1}|_{(z_l, \tau^*)} = 0, \quad \frac{\partial f}{\partial \tau}|_{(z_l, \tau^*)} = 0 \quad (46)$$

by considering (40). The second order partial derivatives of the function f are easily to find out to be

$$\begin{aligned} \frac{\partial^2 f}{(\partial u^1)^2} &= \frac{\partial^2 \zeta^2}{(\partial u^1)^2} + 2 \frac{\partial^2 \zeta^2}{\partial u^2 \partial u^1} f_1^2 + \frac{\partial^2 \zeta^2}{\partial u^2} f_{11}^2 + \frac{\partial^2 \zeta^2}{(\partial u^2)^2} (f_1^2)^2 \\ \frac{\partial^2 f}{\partial u^1 \partial \tau} &= \frac{\partial^2 \zeta^2}{\partial u^1 \partial \tau} + \frac{\partial^2 \zeta^2}{\partial u^2 \partial u^1} f_\tau^2 + \frac{\partial^2 \zeta^2}{\partial u^2 \partial \tau} f_1^2 + \frac{\partial^2 \zeta^2}{\partial u^2} f_{1\tau}^2 + \frac{\partial^2 \zeta^2}{(\partial u^2)^2} f_1^2 f_\tau^2 \\ \frac{\partial^2 f}{\partial \tau^2} &= \frac{\partial^2 \zeta^2}{\partial \tau^2} + 2 \frac{\partial^2 \zeta^2}{\partial u^2 \partial \tau} f_\tau^2 + \frac{\partial^2 \zeta^2}{\partial u^2} f_{\tau\tau}^2 + \frac{\partial^2 \zeta^2}{(\partial u^2)^2} (f_\tau^2)^2 \end{aligned}$$

which at (z_l, τ^*) are denoted by

$$A = \frac{\partial^2 f}{(\partial u^1)^2} \Big|_{(z_l, \tau^*)}, \quad B = \frac{\partial^2 f}{\partial u^1 \partial \tau} \Big|_{(z_l, \tau^*)}, \quad C = \frac{\partial^2 f}{\partial \tau^2} \Big|_{(z_l, \tau^*)}. \quad (47)$$

Then, from (44), (46) and (47), we obtain the Taylor expansion of $f(u^1, \tau)$

$$f(u^1, \tau) = \frac{1}{2}A(u^1 - a_l^1)^2 + B(u^1 - a_l^1)(\tau - \tau^*) + \frac{1}{2}C(\tau - \tau^*)^2$$

which by (43) is the behavior of ζ^2 in the neighborhood of (z_l, τ^*) . Because of the second equation of (32), we get

$$A(u^1 - a_l^1)^2 + 2B(u^1 - a_l^1)(\tau - \tau^*) + C(\tau - \tau^*)^2 = 0$$

which leads to

$$A\left(\frac{du^1}{d\tau}\right)^2 + 2B\frac{du^1}{d\tau} + C = 0 \quad (48)$$

and

$$C\left(\frac{d\tau}{du^1}\right)^2 + 2B\frac{d\tau}{du^1} + A = 0. \quad (49)$$

The solutions of equations (48) or (49) give the different directions of the branch curves at the bifurcation point. The remainder component $du^2/d\tau$ can be given by

$$\frac{du^2}{d\tau} = f_1^2 \frac{du^1}{d\tau} + f_\tau^2$$

where partial derivative coefficients f_1^2 and f_τ^2 have been calculated in (42). Now we get all the different directions of the branch curves. This means the behavior of the electric topological current at the bifurcation points is detailed.

The above solutions reveal the evolution of the electric topological current. Besides the generation and the annihilation of the topological charges, it also includes the splitting and merging of electric topological current. When an original electric topological current moves through the bifurcation point in k -space, it may split into two electric topological currents moving along different branch curves, or, two original electric topological currents merge into one electric topological current at the bifurcation point. The identical conversation of the electric topological current shows the total electric charge must keep invariant before and after the evolution at the bifurcation point, i.e.

$$\beta_{l_1}\eta_{l_1} + \beta_{l_2}\eta_{l_2} = \beta_l\eta_l$$

for fixed l . Furthermore, from above studies, we see that the generation, annihilation and bifurcation of electric topological current are not gradual changes, but start at a critical value of arguments, i.e. a sudden change.

V. SUMMARY AND CONCLUSION

In this paper, with the gauge potential decomposition and the so called ζ -mapping method, we obtain the inner topological structure of electric topological current and its evolution. The topological quantization of electric topological current is gotten. The electric topological current is found to take a form of generalized function $\delta(\zeta)$ $\zeta = (\zeta^1, \zeta^2)$ with ζ^1, ζ^2 are the real part and imaginary part of the magnetic wave function. The Hopf indices β_l and Brouwer degree η_l of the magnetic wave function reveal the inner topological structure of electric topological current, which are also bring to light the effect of vortices with different current-carrying property. When the Jacobian $D(\frac{\zeta}{u}) = 0$, it is shown that there exist the crucial case of branch process. Based on the implicit function theorem and the Taylor expansion, the evolution of the electric topological current is detailed in the neighborhoods of the branch points of ζ -mapping. The branch solutions at the limit points and the different directions of all branch curves at the bifurcation points are calculated out. Because the electric topological current is identically conserved, the total electric charge will keep to be a constant during the evolution. At the limit points case, it means that the topological quantum numbers of the two generated or annihilated electric topological currents must be opposite at the limit point. It can be looked upon as the topological origin of electric topological charge. At the bifurcation case, the total charges of different branch electric topological currents keeps invariant, which is the topological reason of the conservation law.

ACKNOWLEDGMENTS

This research was supported by National Natural Science Foundation of P. R. China.

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